RELATIONS, MAPPINGS AND FUNCTIONS;

A relation is a mathematical statement that mathematically compares any two elements of a given set.

For example;

For a set A = {1,2,3,4,5}

1 is less than 2.

2 is greater than 1.

3 is one more than 2.

5 is two more than 3.

4 is a multiple of 2.

2 is a factor of 4.

PAPYGRAM.

This is a diagrammatical representation of a mathematical relation for a given set of elements.

Example

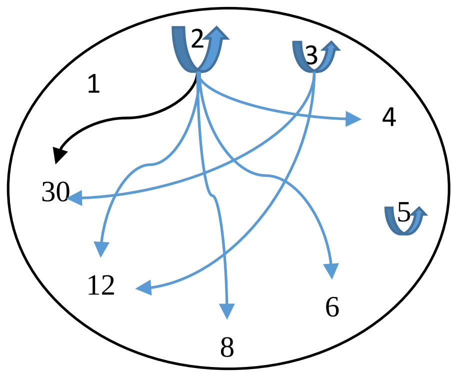
Draw a papygram illustrating the relation "is a prime factor of" in the set {1, 2, 3, 4, 5, 6, 8, 12, and 30}

Solution

In this papygram;

The arrow around an element means that the element is a prime factor of itself.

The other arrows show that the "tail" is a prime factor of the "head"



Exercise

1. Draw a papygram showing the relation "is a multiple of" in the set {42, 28, 21, 14, 7}

2.Given that T = {2, 5, 6, 8, 9, 10, 12, 13} , illustrate on papygrams the relations:

(i) "Greater than by 3"
(ii) "Factor of"

MAPPING

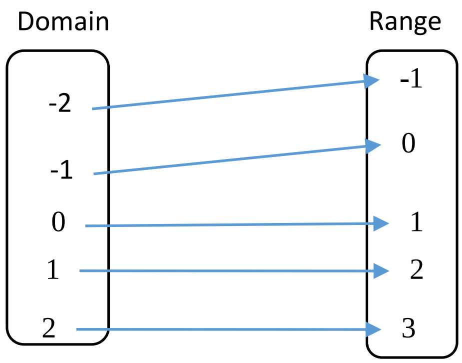
This is a statement that mathematically associates an element of one set with one or more elements of another set.

In a mapping, the set of starting values is called the domain whereas the set of resulting outcomes is called the Range.

Like in a relation, a mapping may be represented on diagram. This diagram is called an **arrow diagram**.

Example;

The figure below shows a mapping of elements in the **domain** to the elements in the **range**. Each element in the domain is increased by 1 to get the corresponding element in the range.



There are four types of mappings;

a) One to one mapping.
b) One to many mapping.
c) Many to one mapping.
d) Many to many mapping.

<u>ONE TO ONE MAPPING</u>

This defines a mapping in which each element in the domain is associated with only one element in the range.

For example; in a mapping x is mapped on to $2x+1$ on the domain; $\{-1,0,1,2\}$.

When $x=-1$ then $2x+1=2(-1)+1$

$$=-1$$

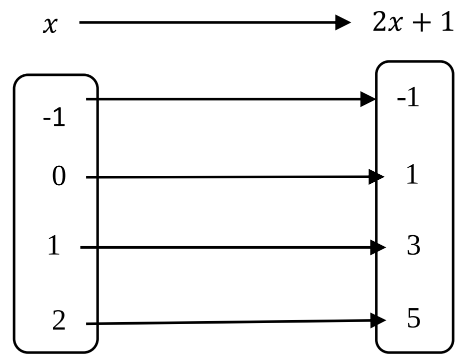
When $x=0$ then $2x+1=1$

When $x=1$ then $2x+1=3$

When $x=2$ then $2x+1=5$

It follows that the Range$=\{-1,1,3,5\}$.

The arrow diagram is as below;



<u>ONE TO MANY MAPPING</u>

This is a mapping in which at least one element of the domain is associated with more than one element in the range.

For example, the mapping x is mapped on to $\sqrt{x}$ on the domain $\{1,4,9\}$.

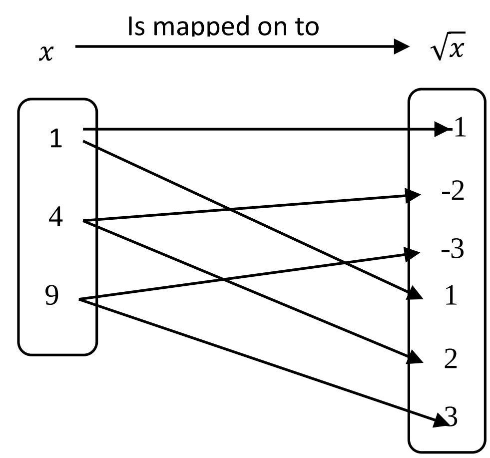
When $x=1$ then $\sqrt{x}=\pm1$

When $x=4$ then $\sqrt{x}=\pm2$

When $x=9$ then $\sqrt{x}=\pm3$

Therefore, the range$=\{-1,-2,-3,1,2,3\}$

The arrow diagram is as below;



<u>MANY TO ONE MAPPING</u>

This defines a mapping in which at least two elements in the domain are associated with a single element in the range.

For example, in the mapping in which x is mapped on to x^2 with the domain = $\{-1,0,1,2\}$.

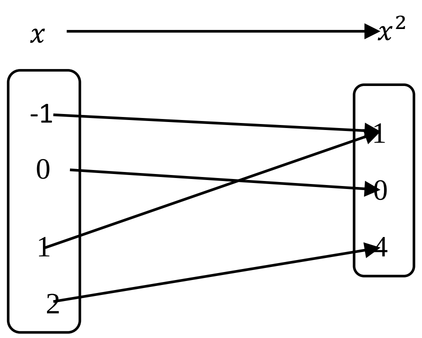
When $x = -1$ then $x^2 = 1$

When $x = 0$ then $x^2 = 0$

When $x = 1$ then $x^2 = 1$

When $x = 2$ then $x^2 = 4$

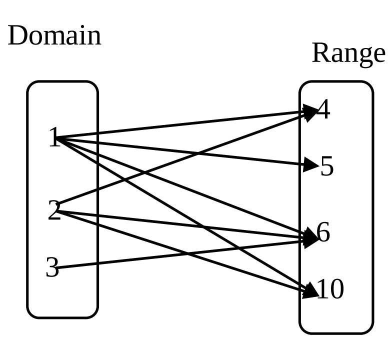
Therefore the Range= $\{0,1,4\}$ and the arrow diagram is as below;



MANY TO MANY MAPPING

This defines a mapping in which at least two elements of the domain are associated with more than one element in the range. For example in the mapping "Is a factor of" on the Domain= $\{1,2,3\}$ and the Range= $\{4,5,6,10\}$.

The arrow diagram is as below;



FUNCTIONS

It should be noted that function is a subset of mappings. A function defines a mapping in which each element in the domain is associated with only and only one element in the Range.

This implies that only the one to one and many to one mappings represent functions. Mappings that represent functions can be written as follows;

a) $x \to 3x + 4$ as a mapping.
b) $f: x \to 3x + 4$, which is read as a function f such that x is mapped on to $3x + 4$
c) $f(x) = 3x + 4$ Which is an algebraic formula that gives the values of the Range given particular values of x in the domain.

Example

1. (i) Determine the range corresponding to the domain $\{0,1,2,3\}$ for the mapping $f(x) = 3x + 1$.

(ii) Represent the mapping in (i) above on an arrow diagram

Solution

(i).When $x = 0$ $f(0) = 3(0) + 1$

$f(0) = 1$

When $x = 1$ $f(1) = 3(1) + 1$

$f(1) = 4$

When $x = 2$ $f(2) = 3(2) + 1$

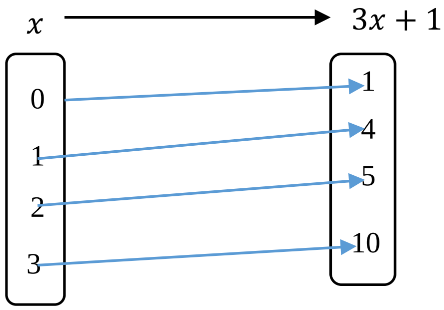
$f(2) = 7$

When $x = 3$ $f(3) = 3(3) + 1$

$f(3) = 10$

Therefore the Range= $\{1,4,7,10\}$ in general $\{f(0), f(1), \ldots\ldots\}$

(ii)



.

Example

A function f(x) is defined by: $f(x) = 4x + 5$, find f(5).

Solution

$f(x) = 4x + 5$

$f(5) = ?$

In order to obtain f(5), we have to substitute the value of x in the expression $4x + 5$ with 5.

$f(5) = 4(5) + 5$

$= 25.$

Note;

A function can be defined using any alphabet however the letters f, g and h are the most commonly used.

Example

Given that $g(x) = \frac{1}{x}$ find

g(a)

g(x + a)

Solution

(i). $g(x) = \frac{1}{x}$

$\therefore g(a) = \frac{1}{a}$

(ii). $g(x + a) = \frac{1}{x+a}$

Example

Given that $h(x) = x^2 + 3x - 9$, find

h(1)

h(−5)

Solution

$h(x) = x^2 + 3x - 9$

(i). $h(1) = (1)^2 + 3(1) - 9$

$= 1 + 3 - 9$

$= -5$

(ii). $h(-5) = (-5)^2 + 3(-5) - 9$

$= 25 - 15 - 9$

$= 1$

Example

A function is defined by the formula $f(x) = 3x - 2$. If $f(a) = 19$, find the value of a.

Solution

$f(x) = 3x - 2$

$f(a) = 3(a) - 2$, but $f(a) = 19$.

$\therefore 3a - 2 = 19$

$3a = 21$

$a = 7.$

Given that; $g(x) = \frac{2}{2x^2-6}$. Find the value of p for which $f(p) = 1$.

Solution

$g(x) = \frac{2}{2x^2-6}$.

$g(p) = \frac{2}{2p^2-6} = 1$

$2p^2 - 6 = 2$

$2p^2 = 8$

$p^2 = 4$

$\therefore p = \pm 2.$

Example

Given that $g(x) = ax^2 + b, g(-2) = 3$ and $g(1) = -3$. Find the value of a and b.

Solution

$g(x) = ax^2 + b$

For $g(-2)$;

$g(-2) = a(-2)^2 + b = 3$

$4a + b = 3$…………………(i)

For $g(1)$;

$g(1) = a(1)^2 + b = -3$

$a + b = -3$…………………(ii).

Equation(i)− equation(ii)

$3a = 6$………………………………(iii)

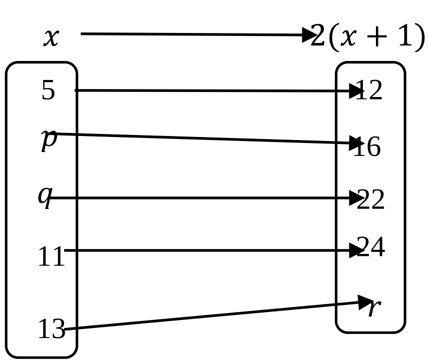
$a = 2$

$2 + b = -3$

$\therefore b = -5.$

Example

Find the unknown values in the arrow diagrams for the given mapping;



Solution

From the diagram, x represents the Domain while $2(x+1)$ represents the Range.

$\therefore f(x) = 2(x+1)$

$f(p) = 2(p+1) = 16$

$2p = 14$

$\therefore p = 7$

$f(q) = 2(q+1) = 22$

$2q = 20$

$\therefore q = 10.$

$r = f(13)$

$r = 2(13+1)$

$\therefore r = 28$

Example

Given that $f(x) = \frac{3x+4}{(x+2)(x-2)} + \frac{2}{x+2}$.Express f(x) in the form $f(x) = \frac{px}{x^2+q}$ hence state the values of p and q.

Solution

$f(x) = \frac{3x+4}{(x+2)(x-2)} + \frac{2}{x+2}$

$f(x) = \frac{3x+4+2(x-2)}{(x+2)(x-2)}$

$f(x) = \frac{5x}{x^2-4}$

For $\frac{px}{x^2+q} = \frac{5x}{x^2-4}$

$p = 5$ and $q = -4$

UNDEFINED FUNCTIONS:

Any function is said to be undefined or meaningless if and only if its denominator is equal to zero. This because when any number or expression is divided by zero, the result is very huge that we cannot define it.

This implies that $f(x) = \frac{g(x)}{h(x)}$ is undefined or meaningless for $h(x) = 0$.

Example

Given that $f(x) = \frac{2}{1-x}$ find

a) $f(-2)$
b) The value of x for which the function is undefined or meaningless.

Solution

(i). $f(x) = \frac{2}{1-x}$

$f(-2) = \frac{2}{1-(-2)}$

$f(-2) = \frac{2}{3}$.

(ii). f(x) is undefined or meaningless if $1 - x = 0$

From $1 - x = 0$

$\therefore x = 1$

Example

Find the value of x for which $f(x) = \frac{5x-6}{4-x^2}$ is not defined.

Solution

f(x) is undefined if, $4 - x^2 = 0$

From $4 - x^2 = 0$

$x^2 = 4$

$x = \pm\sqrt{4}$

$\therefore x = \pm 2$ for f(x) to be undefined.

INVERSE OF A FUNCTION

This defines a function which maps the range of a given function back to the domain. In other words, the inverse of a given function does the opposite of the given function.

For any function f(x) its inverse is usually written as $f^{-1}(x)$. (read as inverse function of f(x)).

If $f(x) = x + 1$ is a given function and we let $f(x) = y$ it follows that; $y = x + 1$.

This means that $f(x) = x + 1$ is an expression that gives the values of y when x is known. The inverse of this function should therefore give the values of x when y is known.

From $y = x + 1$ an expression that gives values of x is obtained by making x the subject of the algebraic formula.

i.e $x = y - 1$ this is the function that does the opposite of $y = x + 1$. And represents the inverse of f(x).

Note;

For convenience the inverse function is written in terms of x or the original variable or unknown.

$\therefore$ For $f(x) = x + 1$ then $f^{-1}(x) = x - 1$

The following examples illustrate how to get the inverse of a given function.

Example

Given that $f(x) = 4x - 8$. Find

$f^{-1}(x)$

$f^{-1}(1)$

Solution

(a). step1: let $f(x) =$ y or any other unknown of your choice.

$y = 4x - 8$……………………(i)

Step2: make x the subject of the resulting algebraic equation

$4x = y + 8$

$\therefore x = \frac{y+8}{4}$..................................(ii)

Step3: replace x with $f^{-1}(x)$ and y with x.

$f^{-1}(x) = \frac{x+8}{4}$ This is the inverse of f(x).

(b). $f^{-1}(1) = \frac{(1)+8}{4}$

$f^{-1}(1) = \frac{9}{4}$.

Example

Obtain the inverse of the function $g(x) = \frac{x+1}{x-1}$, hence find $g^{-1}(-2)$.

Solution

$g(x) = \frac{x+1}{x-1}$,

Let $g(x) = y$

$y = \frac{x+1}{x-1}$ .... (i)

$y(x-1) = x + 1$

$yx - x = y + 1$

$x(y-1) = y + 1$

$\therefore x = \frac{y+1}{y-1}$ .... (ii)

$g^{-1}(x) = \frac{x+1}{x-1}$.

$g^{-1}(-2) = \frac{-2+1}{-2-1}$

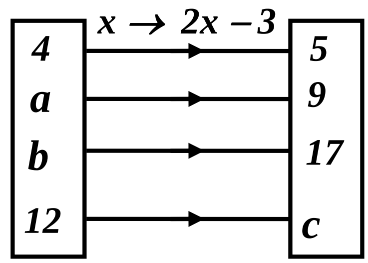
$= \frac{1}{3}$

Exercise

1. (i) Determine the range corresponding to the domain {–3, –2, 0, 1, 2, 3} for the mapping $x \to x^2 + 1$.

   (ii) Represent the mapping in (i) above on an arrow diagram.

2. . Find the unknown values in the arrow diagram for the mapping $x \to 2x - 3$.



3. Given the function $f(x) = \frac{10x}{x^2 - 4}$, find:

   (i) f(3)

   (ii) f(6)

   (iii) the values of x for which f(x) is undefined

   (iv) the values of x for which f(x) = 6

4. Given the function $f(x) = \frac{2x}{3x^2 - 10x - 8}$, find:

   (i) f(2)

   (ii) f(–1)

   (iii) the values of x for which f(x) is undefined

5. Given that f(x) = 3x +5, $g(x) = \frac{2}{2x - 6}$ and $h(x) = \frac{4 - x^2}{x^2 + 3}$, find:

   (i) $f^{-1}(x)$, and hence $f^{-1}(-1)$

   (ii) $g^{-1}(x)$, and hence $g^{-1}(-2)$

   (iii) $h^{-1}(x)$, and hence $h^{-1}(0)$

6. Given that f(x) = ax – 7 and f(8) = 17, find:

(i) the value of a

(ii) f(4)

(ii) $f^{-1}(x)$, hence obtain $f^{-1}(8)$.

7. Given that f(x) = a + bx, f(1) = 8 and f(–1) = 2, find:

(i) the values of a and b

(ii) f(–2)

(iii) f(5)

(iv) the value of c for which f(c) = –7

(v) $f^{-1}(x)$

(vi) $f^{-1}(-7)$

8.Given that f(x) = ax + 9 and $f^{-1}(13) = 1$, find:

(i) the value of a

(ii) $f^{-1}(1)$

9.Given that f(x) = px + 7 and $g^{-1}(x) = \frac{5-2x}{3}$, find:

(i) $g(x)$

(ii) the value of p for which g(2x – 3) = f(x)

10.Given that $g^{-1}(x) = \frac{1+x}{x}$, find $g(3)$.

11. Given that $f(x) = \frac{2}{x+2} + \frac{8x+4}{x^2-4}$, express f(x) in the form $\frac{ax}{x^2+b}$.

Hence find:

(i) f(3)

(ii) the values of x for which f(x) is undefined

12. Given that $f(x) = \frac{2}{3x+2} + \frac{5x+3}{9x^2-4}$, express f(x) in the form $\frac{ax+b}{cx^2+d}$.

Hence find:

(i) f(2)

(ii) the values of x for which f(x) is undefined

13. Given that $f(x) = x^2 - 12$ and $g(x) = 2x - 5$, find:

(i) An expression for gf(x) and hence evaluate gf(4)

(ii) An expression for fg(x) and hence evaluate fg(2)

(iii) The values of x for which gf(x) = fg(x)

(iv) An expression for gg(x) and hence evaluate gg(2)

(v) An expression for ff(x) and hence evaluate ff(–3)

END